Transition from Abelian to Non-Abelian Quantum Liquids in the Second Landau Level

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The search for non-Abelian quantum Hall states in the second Landau level is narrowed to the range of filling factors $7/3 < \nu_e < 8/3$. In this range, the analysis of energy spectra and correlation functions, calculated including finite width and Landau level mixing, supports the prominent non-Abelian candidates at $\nu_e = 5/2$ (Moore–Read) and 12/5 (Read–Rezayi). Outside of it, the four-flux noninteracting composite fermion model is validated. The borderline $\nu_e = 7/3$ state is adiabatically connected to the Laughlin liquid, but its short-range correlations and charge excitations are different.

The incompressible quantum liquids (IQLs) [1] formed in a high magnetic field B by two-dimensional electrons filling various fractions ν of different Landau levels (LL_n, $n = 0, 1, \ldots$) have been the subject of extensive studies ever since the famous discovery of the fractional quantum Hall (FQH) effect [2]. The most recent storm of interest is motivated by the concept of "topological quantum computation" [3, 4] employing non-Abelian statistics of the Moore–Read "pfaffian" wave function [5, 6] believed to describe the FQH state in a half-filled LL_1. Other wave functions with different complexities of braiding statistics have also been proposed [7, 8, 9], elevating the convincing demonstration of non-Abelian statistics in a real physical system to the challenge of greatest current importance.

With the partially filled LL_0 successfully described by the composite fermion (CF) theory [10] and with ordered electron phases favored in higher LLs, signatures of non-Abelian statistics are most strongly anticipated in the few known IQLs in LL_1. Crucial recent experiments in this LL include confirmation of the quasiparticle (QP) charge of $\frac{1}{4}e$ for the half-filled state at $\nu_e = \frac{5}{2}$ [14] and careful measurements of the minute excitation gaps [11, 12, 13]. In theory, the most recent advances have been related to the breaking of particle-hole symmetry in the Moore–Read state [15], the role of layer width [16] and LL mixing [17, 18] in real systems, and the nature of QPs [19, 20].

Despite intensive studies, connection of the FQH states in LL_1 ($\nu_e = \frac{5}{2}$, $\frac{7}{3}$, $\frac{12}{5}$, or $\frac{11}{5}$) to the few proposed wave functions is not conclusively established, and in some cases it is only tentatively assumed for the lack of other candidates. This is an urgent problem, as the connection of some of these wave functions to the particular conformal field theories is precisely what fuels the anticipation of non-Abelian statistics in nature. Another unresolved puzzle is the discrepancy [13] between the experimental and numerical gaps at $\nu_e = \frac{5}{2}$, which undermines the understanding of this state in simple models assuming spin polarization or decoupling from the crystal lattice. On the other hand, a wealth of IQLs found in various systems (electrons or CFs at different fillings of different LLs, in layers of varied width w) invites a more general question of possible IQLs with arbitrary interactions V.

In this Letter we demonstrate (numerically) that non-Abelian IQLs can only emerge in LL_1 in the narrow range of filling factors $\frac{7}{3} < \nu_e < \frac{8}{3}$. In this range, the known non-Abelian candidates at $\nu_e = \frac{5}{2}$ and $\frac{12}{5}$ are closely examined (including finite layer width and LL mixing) and found to have favorable correlation energies and suggestive excitation gaps. Outside of it, the Abelian ground states of noninteracting CFs carrying four flux quanta repeat in both lowest LLs. The borderline $\nu = \frac{1}{3}$ ground state is adiabatically connected to the Laughlin state, but it has a smaller gap, distinct short-range correlations, negative quasihole (QH) energy in narrow layers, and probably different quasielectrons (QEs).

We study spin-polarized N-particle systems on a unit sphere with the magnetic monopole of strength $2Q(hc/e)$ inside [21]. In this geometry, LL_n is a shell of angular momentum $\ell = Q + n$, and different N-body wave functions at the same ν are distinguished by a 'shift' γ between LL degeneracy and $\nu^{-1}N$. In contrast to previous exact diagonalization calculations, we do not confine ourselves to the systems with particular V, but search for the universality classes (ν, γ) of gapped ground states with arbitrary interactions, attainable in realistic systems (of electrons or CFs) with the suitable choice of n and w.

We begin by recalling that many-body dynamics in a degenerate LL is determined by an interaction pseudopotential, defined [21] as the dependence of the pair energy V on the relative angular momentum $m = 1, 3, \ldots$. Moreover, V_m induces particular correlations through its deviation from a straight line over any consecutive m's [22]. Hence, it is known that low-energy spectra of V_m are reproduced by a suitable effective pseudopotential U_m with only a few (positive or negative) coefficients.

We have used $U = [U_1, U_3, U_5]$ in the search for IQLs with (nearly) arbitrary interactions. Higher terms (causing long-range order) were ignored. On the other hand, inclusion of U_5 was needed for an accurate description of the two lowest (electron or CF) LLs known to host IQL states, while it still allowed for useful graphical representation of the ground state properties in an effectively two-dimensional space of (normalized) parameters U_m.

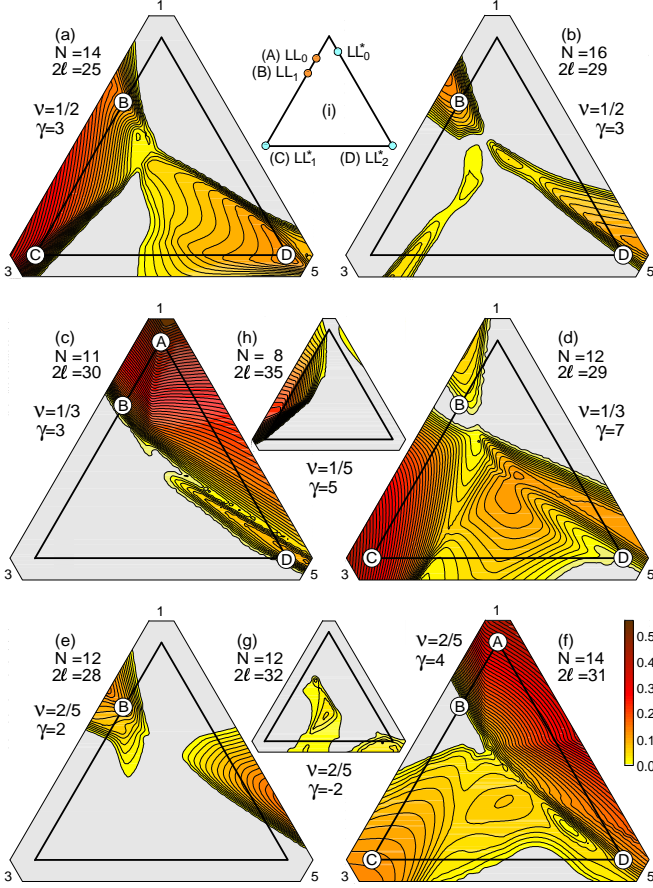We looked at various finite systems $(N, 2\ell)$ in search of

FIG. 1: (color online) Ternary contour plots of the neutral
excitation gaps Δ for N fermions in a partially filled Landau
level with shell angular momentum ℓ, interacting via model
pseudopotential U_m. In each plot, three corners of the big
triangles correspond to $U_m = \delta_{m,\mu}$ with $\mu = 1, 3, 5$ marked
in each corner. Points relevant for the actual interactions in
different electron or CF Landau levels are marked on a trian-
gle in frame (i). Different candidate incompressible states are
indicated in frames (a)–(h), labeled by filling factors ν and
shifts γ assigned to each finite system $(N, 2\ell)$. For example,
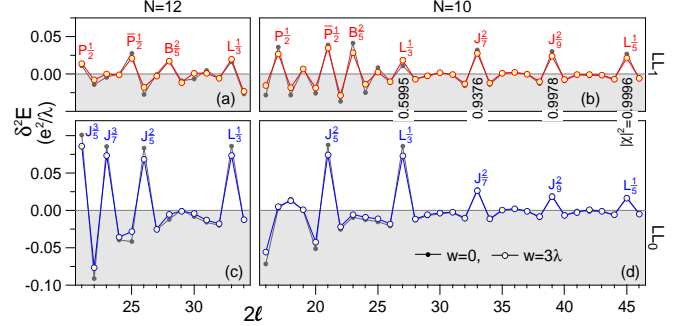the Moore–Read state is marked as (B) in frames (a) and (b).

FIG. 2: (color online) Symmetric second-order differences $\delta^2 E$
of the ground-state energy per particle E for $N = 10$ and 12
electrons in LL_0 and LL_1, as a function of shell angular mo-
mentum ℓ, for layer widths $w = 0$ and 3λ. Candidate incom-
pressible states are marked as '$\Phi\nu$', where $\Phi =$ P, $\overline{P}$, L, J, B
denote Moore–Read pfaffian, anti-pfaffian, Laughlin and Jain
states, and the state of Ref. [9]. Squared overlaps between
the states repeating in both LLs at $\nu \le \frac{1}{3}$ are indicated.

the series of gapped ground states at $2\ell = \nu^{-1}N - \gamma$. A
few maps of the 'neutral' gap Δ (gap to the first excited
state in the same spectrum) appear in Fig. 1. The IQL
candidates are the maxima in Δ repeating regularly for
different N. Their location on the map must be compared
with the actual pseudopotentials in different LLs, whose
approximate positions (A)–(D) are indicated in frame (i).

Let us begin with a half-filled LL represented by $2\ell =
2N - 3$. The maximum near $U_{(B)} = [\frac{2}{3}, \frac{1}{3}, 0]$ found for
each N is relevant for the electrons in LL_1, whose actual
Coulomb pseudopotential is almost linear between $m = 1$
and 5. The maximum coincides with a peak in similar
maps (not shown) of the squared overlap of the ground
state of U with the exact Moore–Read state, and also
with an essentially zero-level minimum of the triplet Hal-

dane amplitude [23]. This confirms quite definitively the
earlier expectations that the Moore–Read ground state
indeed occurs for a class of pseudopotentials close to that
of LL_1 and that its accuracy depends sensitively on the
fine-tuning of the leading V_m's, achieved by adjusting the
layer width w [16]. Remarkably, it precludes the Moore–
Read state in other half-filled LLs (e.g., in the second CF
LL, called LL_1^*, with dominant repulsion at $m = 3$ [24],
where the observed FQH state [25] must be different).

In similar maps for $\nu = \frac{1}{3}$, significant gaps are found in
different areas for the series of states with $\gamma = 3$ (Laugh-
lin) and 7. The latter was proposed earlier for both LL_1
[22] and LL_1^* [26] but, despite numerical evidence for pair-
ing, its wave function remains unknown. It is clear from
Fig. 1(c) and (d) that in the vicinity of $U_{(B)}$, relevant for
LL_1, competition between the $\gamma = 3$ and 7 states must
be (and will be, in the following) resolved more carefully.

For $\nu = \frac{2}{5}$, Jain's series with $\gamma = 4$ correctly represents
the ground state in LL_0. In LL_1, this series competes
with two others: the parafermion $\gamma = -2$ state [7] and
a recently proposed [9] state with $\gamma = 2$. Especially for
the latter, Fig. 1(e) appears suggestive of a gap emerg-
ing around $U_{(B)}$. The subsequent careful analysis of the
competition between these three universality classes is
crucial, because the candidate states with $\gamma = \pm 2$ are
both non-Abelian, in contrast to the $\gamma = 4$ Jain state.

Finally, at $\nu = \frac{1}{5}$, any positive $U = [U_1, U_3, 0]$ yields an
exact Laughlin state. Fig. 1(h) shows that it is true de-
scription of the FQH states in both LL_0 and LL_1 [27]. On
the other hand, its relevance to the FQH effect observed
in LL_1^* [25] is doubtful (indeed, an alternative series with
$\gamma = 9$ appears to have more favorable correlations).

Guided by the maps of Δ, pair/triplet amplitudes, and
various overlaps we now focus on the FQH states in LL_1.
In Fig. 2 we seek confirmation of the IQL candidates in
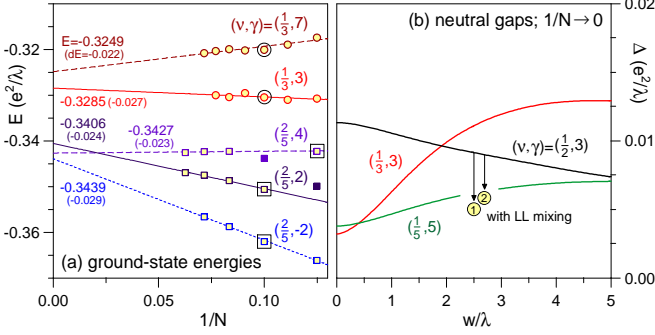cusps of the dependence of the ground-state energy per

FIG. 3: (color online) (a) Size extrapolation of the ground-state energies per particle E calculated for N electrons in the second Landau level. Competing series of candidate incompressible states with filling factors $\nu = \frac{1}{3}$ and $\frac{2}{5}$ are distinguished by their shifts γ. Energy corrections dE due to Landau level mixing were obtained for $B = 2.6$ T and $N = 8$ or 10, as indicated with large open symbols. (b) Dependence of the extrapolated neutral gaps Δ on the layer width w, for the $\nu = \frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{5}$ nondegenerate ground states in the second Landau level. For $\nu = \frac{1}{2}$, circles (1) and (2) show gap reduction due to Landau level mixing relevant for the experimental Refs. [13] and [11], respectively.

particle E on the LL degeneracy. The cusps are most pronounced in the plots of $\delta^2 E_{2\ell} = E_{2\ell-1} + E_{2\ell+1} - 2E_{2\ell}$. For an IQL, its (positive) value gives the QP gap $\tilde{\Delta}$ times the number of QPs created per flux quantum.

Besides complementing the maps of Fig. 1 in the identification of the universality classes of particular IQLs, Fig. 2 reveals a simple connection between statistics and filling factor in LL_1. At $\nu < \frac{1}{3}$ the same Laughlin/Jain IQLs occur in LL_1 and LL_0, with high overlaps and similar gaps. This similarity, earlier pointed out in Ref. [27], is caused by sufficiently high V_1 and similar $V_{m\geq3}$ in both LLs. It validates the noninteracting CF model [10] with four flux quanta attached to each electron in LL_1.

In contrast, at $\nu > \frac{1}{3}$ the Jain sequence of LL_0 is replaced in LL_1 by a different set of IQLs, including the pfaffian and anti-pfaffian pair at $\nu = \frac{1}{2}$ and, apparently, a different $\nu = \frac{2}{5}$ state with $\gamma = 2$ (Fig. 2 shows no signatures of the parafermion $\nu = \frac{2}{5}$ series with $\gamma = -2$ [7]). The breakdown of the noninteracting two-flux CF model in LL_1 opens, exclusively at $\frac{1}{3} < \nu < \frac{2}{3}$, a possibility for other IQLs, including several suggested more exotic states with various non-Abelian statistics of their QPs.

For the borderline $\nu = \frac{1}{3}$ state, separating the Abelian from (possibly) non-Abelian states, our calculation reinforces the theory of Ref. [27]. This state has a moderate overlap with the Laughlin ground state of LL_0, despite falling into the same class of $\gamma = 3$. It also has a smaller gap than the other Laughlin/Jain states in LL_0 or LL_1.

Having narrowed the search for non-Abelian IQLs in LL_1 to the range of $\frac{1}{3} < \nu < \frac{2}{3}$, let us now decide more conclusively between the competing candidate wave functions for the particular filling factors. For $\nu = \frac{1}{2}$, the

issues of the interplay of pfaffian and anti-pfaffian states [15] and of spin polarization [28] were already discussed. In Fig. 3(a) we compare the ground-state energies per particle E of three known candidates for $\nu = \frac{2}{5}$ (we used $N \leq 16$ and $w = 0$; results for $w > 0$ are similar; energies E are rescaled by $\sqrt{2Q\nu/N}$ to ensure equal units e^2/λ for each γ). In the extrapolation we only used the open squares, discarding one state aliased with the anti-pfaffian and one apparently suffering from the small size.

Although Fig. 3(a) suggests that the ground state at $\nu = \frac{2}{5}$ might have $\gamma = -2$ (despite the lack of significant gaps in Figs. 1 and 2), the extrapolated energies of all three competing states are indecisively close. Hence, comparison of their susceptibility to LL mixing is crucial. We estimated the appropriate energy correction dE by including in the diagonalization additional states involving a single cyclotron excitation [17]. For each γ, the value of dE was calculated for $N \leq 10$ (it was also checked that dE is less N-dependent than E). The assumed Coulomb-to-cyclotron energy ratio $\beta \equiv (e^2/\lambda)/(\hbar\omega_c) = 1.56$ corresponds to $B = 2.6$ T. Clearly, in addition to having the lowest E, the $\gamma = -2$ parafermion state also has the largest $|dE|$, and therefore it is convincingly predicted to define the universality class of the $\nu = \frac{2}{5}$ ground state in LL_1. This is important because this state is the only known candidate IQL [4] whose braiding rules are sufficiently complex to allow quantum computation.

Fig. 3(a) presents also the energies for $\nu = \frac{1}{3}$. Here, the $\gamma = 7$ series is firmly ruled out in favor of the $\gamma = 3$ states (adiabatically connected to the Laughlin state).

In Fig. 3(b) we plot the extrapolated neutral gaps $\Delta(w)$ for several IQLs in LL_1. Unlike in LL_0 and in agreement with the experiments [11, 12], the gaps at $\nu = \frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{5}$ are all similar, and the gap at $\nu = \frac{2}{5}$ is much smaller (not shown). Understanding that Δ includes the QE–QH attraction, it is nevertheless noteworthy that incorporating an earlier estimate of gap reduction due to LL mixing at $\nu = \frac{1}{2}$ [17] gives a surprisingly good agreement with experiments in the high mobility limit: for Refs. [11, 12], $w/\lambda = 2.7$ and $\beta = 1.1$, yielding the 35% reduction and $\Delta = 0.006\, e^2/\lambda$; for Ref. [13], $w/\lambda = 2.5$ and $\beta = 1.7$, yielding the 45% reduction and $\Delta = 0.005\, e^2/\lambda$.

Let us now examine more closely the crossover $\nu = \frac{1}{3}$ ground states in LL_1. For $N \leq 12$, their squared overlaps with the Laughlin state are merely $\sim 50\%$. This reflects quite different correlations (e.g., for $N = 12$ and $w = 3\lambda$, the pair amplitudes at $m = 1, 3, 5, 7$ are 0.02, 0.13, 0.17, 0.10 in LL_1, as opposed to 0.00, 0.18, 0.13, 0.09 in LL_0). The magneto-roton band is also absent in the spectra of LL_1. The low-energy states resembling Laughlin QEs and QHs are found at $2\ell = 3N - 3 \mp 1$, but they are not generally the lowest states in their spectra. For example, the ground state at $2\ell = 3N - 4$ has angular momentum $L = 2, \frac{3}{2}, 1, \frac{1}{2}, 0$ for $N = 8, \ldots, 12$ (instead of $L = \frac{1}{2}N$).

Although we have checked that the eigenstates adiabatically connected to the Laughlin QPs become the low-
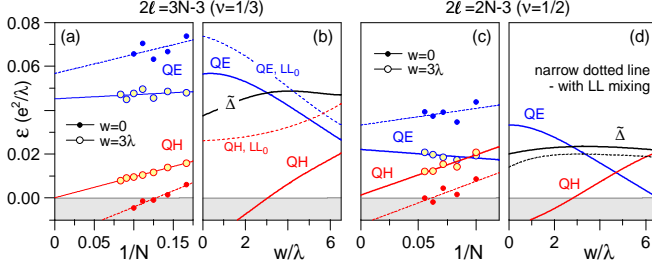
FIG. 4: (color online) Quasiparticle energies ε at the filling factors $\nu = \frac{1}{3}$ and $\frac{1}{2}$ in the second Landau level: (a,c) Size extrapolation of the N-electron results for the layer widths $w = 0$ and 3λ. (b,d) Extrapolated quasielectron and quasihole energies, and their sum $\tilde{\Delta}$, as a function of width w. In (b), curves for the lowest Landau level are added for comparison. In (d), dotted line includes Landau level mixing ($B = 2.6$ T).

est states under bias, their joint energy $\tilde{\Delta} = \varepsilon_{\mathrm{QE}} + \varepsilon_{\mathrm{QH}}$ is unreasonably high. Fig. 4(a,b) shows the size extrapolation of QE and QH energies ε, and width dependences of the extrapolated ε's. Remarkably, $\varepsilon_{\mathrm{QH}} > 0$ requires finite width $w > 3\lambda$. Negative QH energies in narrow layers are also found in other states in LL_1 identified in Fig. 2(b). For instance, $\varepsilon = \frac{1}{2}(E_{2\ell\pm1} - E_{2\ell})$ plus the size corrections [29] are plotted in Fig. 4(c,d) for $\nu = \frac{1}{2}$.

Note that the gap $\tilde{\Delta}$ at $\nu = \frac{1}{2}$ is not immediately weakened in wider wells (in contrast to Ref. [29] where only data for $N = 10$ and 14 was used for $w > 0$). The initial gap enhancement in wider wells is even more pronounced with the inclusion of LL mixing (here, the corrections dE were calculated for $N \leq 14$ at $2\ell = 2N - 3$ and $2N - 3 \pm 1$). While the reasonable match between the experimental gaps and Δ (rather than $\tilde{\Delta}$) at $\nu = \frac{1}{2}$ remains a mystery, the difference between the latter two is well understood as the QE–QH atraction [29]. However, the difference between Δ and $\tilde{\Delta}$ at $\nu = \frac{1}{3}$ is too large to be explained in this way. Instead, Laughlin QEs are unlikely to be the elementary negative carriers in LL_1.

In conclusion, we have found the range of filling factors $\frac{1}{3} < \nu < \frac{2}{3}$ in LL_1 in which the emergence of non-Abelian statistics is possible. Inside this range, our calculations including finite layer width and LL mixing demonstrate that the spin-polarized ground states at $\nu = \frac{1}{2}$ and $\frac{2}{5}$ are described by the "pfaffian" and "parafermion" wave functions (both non-Abelian). Outside, the Jain states of noninteracting CFs repeat in both lowest LLs, precluding more exotic phases. The borderline $\nu = \frac{1}{3}$ state is adiabatically connected to the Laughlin liquid but has a smaller gap and distinct excitations.

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